# אינפי 1

## פרק 21 - תרגילים מתקדמים נוספים (הפרק באנגלית)

### תוכן העניינים


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# Convergence of a Sequence, Monotone Sequences (סדרות)

## Questions

**1)** Let $A$ be a non-empty subset of $\mathbb{R}$ and $\alpha = \inf A$. Show that there exists a sequence $(a_n)$ such that an $a_n \in A$ for all $n \in \mathbb{N}$ and $a_n \to \alpha$.

**2)** Let $A$ be a non-empty subset of $\mathbb{R}$ and $x_0 \in \mathbb{R}$. Show that there exists a sequence $(a_n)$ in $A$ such that $|x_0 - a_n| \to d(x_0, A)$. Recall that $d(x, A) = \inf\{|x-a| : a \in A\}$.

**3)** Let $(a_k)$ be a bounded sequence. For every $n \in \mathbb{N}$, define $x_n = \sup\{a_k : k < n\}$. Show that the sequence $(x_n)$ converges.

### Cauchy Criterion, Bolzano - Weierstrass Theorem

**4)** Show that a sequence $(x_n)$ of real numbers has no convergent subsequence if and only if $|x_n| \to \infty$.

**5)** Let $(x_n)$ be a sequence in $\mathbb{R}$ and $x_0 \in \mathbb{R}$. Suppose that every subsequence of $(x_n)$ has a subsequence converging to $x_0$. Show that $x_n \to x_0$.

**6)** Let $(x_n)$ be a sequence in $\mathbb{R}$. We say that a positive integer $n$ is a peak of the sequence if $m > n$ implies $x_n > x_m$ (i.e., if $x_n$ is greater than every subsequent term in the sequence).

a) If $(x_n)$ has infinitely many peaks, show that it has a decreasing subsequence.

b) If $(x_n)$ has only finitely many peaks, show that it has an increasing subsequence.

c) From (a) and (b) conclude that every sequence in $\mathbb{R}$ has a monotone subsequence. Further, every bounded sequence in $\mathbb{R}$ has a convergent subsequence (An alternate proof of Bolzano-Weierstrass Theorem).


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# Continuity and Limits (גבולות ורציפות)

## Questions

**1)** Let $\lim\limits_{x\to 0}\frac{f(x)}{x^2}=5$. Show that $\lim\limits_{x\to 0}\frac{f(x)}{x}=0$.

**2)** Let $f:\mathbb{R}\to\mathbb{R}$ and $x_0\in\mathbb{R}$. Suppose $\lim\limits_{x\to x_0}f(x)$ exists.
Show that $\lim\limits_{x\to 0}f(x+x_0)=\lim\limits_{x\to x_0}f(x)$.

**3)** Let $f(x)=|x|$ for every $x\in\mathbb{R}$. Show that $f$ is continuous on $\mathbb{R}$.

**4)** Let $f:[0,\pi]\to\mathbb{R}$ be defined by $f(0)=0$ and $f(x)=x\sin\frac{1}{x}-\frac{1}{x}\cos\frac{1}{x}$ for $x\neq 0$.
Is $f$ continuous?

**5)** Let $[\cdot]$ denote the integer part function and $f:[0,\infty)\to\mathbb{R}$ be defined by
$f(x)=[x^2]\sin\pi x$.

a) Show that $f$ is continuous at each $x\neq\sqrt{n}$, $n\in\mathbb{N}$. [Here $\mathbb{N}$ includes 0]
b) Show that $f$ is continuous at each $x=k\in\mathbb{N}$.
c) Show that $f$ is discontinuous at each $x=\sqrt{n}$, $n\in\mathbb{N}$ such that $x\notin\mathbb{N}$.

**6)** Let the function $f:[0,1]\to[a,b]$ be one-one and onto. Suppose $f$ is continuous.
Show that $f^{-1}$ is also continuous.

**7)** Let $f:(0,1)\to\mathbb{R}$ be given by

$$f(x)=\begin{cases}\frac{1}{q} & \text{if } x=\frac{p}{q} \text{ where } p,q\in\mathbb{N} \text{ and } p,q \text{ have no common factor}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$$

a) Suppose $x_n\to x_0$ for some $x_0$, with $x_n\neq x$ for all $n\in\mathbb{N}$, and suppose $x_n=\frac{p_n}{q_n}\in(0,1)$ where $p_n,q_n\in\mathbb{N}$ have no common factors. Show that $\lim\limits_{n\to\infty}q_n=\infty$.
b) Show that $f$ is continuous at every irrational.
c) Show that $f$ is discontinuous at every rational.


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# Existence of Extrema, Intermediate Value Property (משפט ערך הביניים ומשפט ויירשטראס)

## Questions

**1)** Give an example of a function $f$ on $[0,1]$ which is not continuous but satisfies the IVP*. *We say that $f$ has the property IVP [Intermediate Value Property] on $[a,b]$ if for every $x,y\in[a,b]$ and $\alpha$ satisfying $f(x)<\alpha<f(y)$ or $f(x)>\alpha>f(y)$ there exists $x_0\in[x,y]$, such that $f(x_0)=\alpha$.

**2)** Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. Show that $f$ is a constant function if
   a) $f(x)$ is rational for each $x\in\mathbb{R}$.
   b) $f(x)$ is an integer for each $x\in\mathbb{Q}$.

**3)** Let $p(x):\mathbb{R}\to\mathbb{R}$ be a polynomial function of odd degree. Show that $p$ is onto.

**4)** Let $f,g:[0,1]\to\mathbb{R}$ be continuous such that
$\inf\{f(x):x\in[0,1]\}=\inf\{g(x):x\in[0,1]\}$.
Show that there exists $x_0\in[0,1]$ such that $f(x_0)=g(x_0)$.

**5)** A cross country runner runs continuously an eight kilometers course in 40 minutes without taking rest. Show that, somewhere along the course, the runner must have covered a distance of one kilometer in exactly 5 minutes.

**6)** Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function.
   a) Suppose $f$ attains each of values exactly two times. Given:
   $f(x_1)=f(x_2)=\alpha$ for some $x_1,x_2,\alpha\in\mathbb{R}$, and $f(x_0)>\alpha$ for some $x_0\in[x_1,x_2]$. Show that $f$ attains its maximum in $[x_1,x_2]$ exactly at one point.
   b) Using (a) show that $f$ cannot attain each of its values exactly two times.


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# Mean Value Theorem, L'Hôpital's Rule, Differentiability (משפט לגראנז', כלל לופיטל וגזירות)

## Questions

**1)** Does there exist a differentiable function $f:[0,2]\to\mathbb{R}$ satisfying $f(0)=-1,\ f(2)=4$ and $f'(x)\le 2$, for all $x\in[0,2]$?

**2)** Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable such that for some $\alpha\in\mathbb{R}$, $|f'(x)|\le\alpha<1$ for all $x\in\mathbb{R}$. Let $a_1\in\mathbb{R}$ and define a sequence $(a_n)$ recursively by $a_{n+1}=f(a_n)$. Show that $(a_n)$ converges.

**3)** Let $f:[a,b]\to\mathbb{R}$ be differentiable and let $\alpha\in\mathbb{R}$ be such that $f'(a)<\alpha<f'(b)$. Define $g(x)=f(x)-\alpha x$ for all $x\in[a,b]$.

a) Show that there exists $c\in[a,b]$ such that $g'(c)=0$.
Hint: prove by contradiction, noting that $g'(a)<0$ and $g'(b)<0$.

b) From the above, conclude that if a function $f$ is differentiable on an interval $[a,b]$, then $f'$ has the Intermediate Value Property on $[a,b]$.

**4)** Suppose $f:[0,1]\to\mathbb{R}$ is continuous and $\int_0^1 f(t)dt=1$.

a) Show that there exists $c\in(0,1)$ such that $f(c)=1$.

b) Show that there exist $c_1\neq c_2$ in $(0,1)$ such that $f(c_1)+f(c_2)=2$.

**5)** Let $f:[0,1]\to\mathbb{R}$ be such that $|f'(x)|<10$ for all $x\in(0,1)$ and let $(x_n)$ be a sequence in $(0,1)$ satisfying the Cauchy criterion. Show that the sequence $(f(x_n))$ converges.

**6)** Let $f:[0,1]\to\mathbb{R}$ and $a_n=f\left(\frac{1}{n}\right)-f\left(\frac{1}{n+1}\right),\ n=1,2,...$
Show that:

a) if $f$ is continuous, then $\sum_{n=1}^{\infty}a_n$ converges;

b) if $f$ is differentiable and $|f'(x)|<\frac{1}{2}\ \forall x\in[0,1]$, then $\sum_{n=1}^{\infty}a_n(\cos n)\sqrt{n}$ converges.


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**7)** Let $p(x)=a+b\,x+cx^2$. Find all values of $a,b,c\in\mathbb{R}$ for which the function $p(|x|)$ is differentiable at 0.


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# Power Series, Taylor Series (טורי חזקות וטורי טיילור)

## Questions

**1)** Let $f:(a,b)\to\mathbb{R}$ be infinitely differentiable and let $x_0\in(a,b)$. Suppose that there exists $M>0$ such that $\left|f^{(n)}(x)\right|\le M^n$ for all $n\in\mathbb{N}$ and $x\in(a,b)$.
Show that Taylor's series of $f$ around $x_0$ converges to $f(x)$ for all $x\in(a,b)$.

**2)** Let $(a_n)$ be a sequence of nonnegative reals and suppose that $(a_n^{\frac{1}{n}})$ is a bounded sequence. For each $n$, define $A_n=\sup\{a_k^{\frac{1}{k}}:k\ge n\}$.
$(A_n)$ converges since it is decreasing and bounded below (by 0).
So $A_n\to L$ for some $L\ge 0$.

a) Show that if $L<1$, the series $\sum_{n=1}^{\infty}a_n$ converges and if $L>1$ the series diverges.

b) Show that the radius of convergence of the power series $\sum_{n=1}^{\infty}a_nx^n$ is $\frac{1}{L}$.


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